# Ab initio results for the broken phase of scalar light front field theory

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(Dated: May 5, 2005)

# Abstract

We present nonperturbative light-front energy eigenstates in the broken phase of a two dimensional $\frac{\lambda}{4!}\phi^4$ quantum field theory using Discrete Light Cone Quantization and extrapolate the results to the continuum limit. We establish degeneracy in the even and odd particle sectors and extract the masses of the lowest two states and the vacuum energy density for $\lambda = 0.5$ and $1.0$. We present two novel results: the Fourier transform of the form factor of the lowest excitation as well as the number density of elementary constituents of that state. A coherent state with kink - antikink structure is revealed.

PACS numbers: 11.10Ff, 11.10 Kk, 11.10Lm, 11.30Qc

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Quantum descriptions of spontaneous symmetry breaking (SSB) and topological objects play a major role in condensed matter physics, quantum field theory and string theory. Since a direct solution of the quantum problem is extremely difficult, traditional approaches rely on semi-classical expansions, variational approximations like the coherent state ansatz, self-consistent methods like the Hartree approximation, etc. Using these methods, one can test for the presence of SSB and compute the leading quantum corrections to topological objects in a model. One would like to know the reliability of these approximation schemes, to calculate the masses nonperturbatively and also to determine additional observables from Hamiltonian eigenfunctions. For example, with the eigenfunctions, one could study whether the number density of "elementary constituents" follows the behavior suggested by the variational coherent state ansatz.

We will use the Discrete Light Cone Quantization method (DLCQ) [1] to address these issues. This method can provide exact (nonperturbative) predictions, in the continuum limit, of the masses of the lowest states and vacuum energy density.

In the light front literature it has been suggested that (see the review in Ref. [1]) without a field mode carrying exactly zero momentum, it would be impossible to describe spontaneous symmetry breaking. Rozowsky and Thorn [2] pointed out that if such a mode were really necessary, one of the attractive features of light front formulations of field theory and string theory, namely, casting any relativistic quantum mechanical system in a Newtonian picture, would no longer be viable. They convincingly argued that the physics of condensation does not require a mode with exactly zero momentum. They performed an analytic variational calculation at weak coupling to corroborate their hypothesis and we utilize their coherent state ansatz to help interpret our results. However, the numerical calculations carried out by these authors to independently verify this hypothesis were not conclusive since the volumes considered were not large enough to overcome tunneling.

One of our main results is that we do observe SSB even without the explicit presence of the notorious zero-momentum mode as anticipated by Rozowsky and Thorn [2]. We also compute the Fourier transform of the form factor of the lowest state (i.e., the " profile" of the kink-antikink configuration) and its parton content, results not yet available from other methods. In addition, at weak coupling, we extract the value of the condensate from two observables, (1) the computed vacuum energy density, and (2) the asymptote of the profile function.

As is well known [3, 4], topological excitations (kinks) exist in the classical as well as quantum two-dimensional $\phi^4$ model with negative quadratic term (broken phase). It was proven rigorously that in quantum theory a stable kink state is separated from the vacuum by a mass gap of the order $\lambda^{-1}$ and from the rest of the spectrum by an upper gap [5]. More detailed nonperturbative information on the spectrum of the mass operator or on other observables from rigorous approaches is not available.

A popular nonperturbative numerical approach to field theory is the Euclidean lattice formulation. In the topologically non-trivial sector of two-dimensional $\phi^4$ theory, results available from lattice simulations, so far [6], are limited to the determination of the kink mass. The results for the configuration average of the kink profile are not smooth and are difficult to interpret, perhaps due to problems with thermalization and/or finite volume limitations.

These Euclidean lattice calculations are highly non-trivial and a brief overview displays the degree of effort needed to reveal topological observables. In one approach, one computes the kink mass from the decay of the correlation functions of an operator with nonvanishing projection on the topological sector under consideration, the dual field in the present case. On a finite lattice, the definition of such an operator is often ambiguous. Another approach involves integrating the difference between the expectation values of the lattice actions with antiperiodic and periodic boundary conditions. To obtain results for the continuum field theory, one has to work in the critical region of the lattice theory. Here, calculations are severely hampered by the phenomena of critical slowing down. Given these difficulties, it is understandable that Euclidean lattice calculations of the mass and other properties of the kink-antikink state have not been reported to date. The strengths of the Euclidean lattice approach lie in extracting critical properties and low mass eigenstates of the theory. On the other hand, the strengths of DLCQ are complementary and lie in extracting a range of observables for the mass eigenstates.

In the variational approach, kinks can be well-approximated by coherent states. This appears to have two implications for the Fock space expansion in our discretized approach, namely, (a) one may need an infinite number of bosons to describe solitons, and, (b) since the dimensionless total longitudinal momentum $K$ automatically provides a cutoff on the number of bosons, convergence in $K$ may be difficult to achieve for a kink-like state [7]. Here, we show how a nonperturbative evaluation of topological excitations and their observables

is feasible in a finite Fock basis.

We investigate these issues in two dimensional $\phi^4$ theory using DLCQ with periodic boundary condition (PBC). We adopt the convention $x^{\pm} = x^0 \pm x^1$. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1}$$

The Hamiltonian density $\mathcal{P}^- = -(1/2)\mu^2\phi^2 + (\lambda/4!)\phi^4$ and the momentum density $\mathcal{P}^+ = \partial^+\phi\partial^+\phi$. In DLCQ, the Hamiltonian operator with dimension of $mass^2$, $H = (2\pi/L)P^- = (2\pi/L)\int_{-L}^{+L} dx^- \mathcal{P}^-$ and the dimensionless momentum operator $K = (L/2\pi)P^+ = (L/2\pi)\int_{-L}^{+L} dx^- \mathcal{P}^+$. Here $L$ denotes our compact domain $-L \le x^- \le +L$. Throughout this work we address the eigen solutions of $H$ with $\mu^2 = 1$.

The scalar field can be decomposed as $\phi(x^-) = \phi_0 + \Phi(x^-)$, where $\phi_0$ is the zero mode operator omitted in our Hamiltonian and $\Phi(x^-)$ is the normal mode operator

$$\Phi(x^-) = \frac{1}{\sqrt{4\pi}}\sum_n \frac{1}{\sqrt{n}}\left[a_n e^{-i\frac{n\pi}{L}x^-} + a_n^{\dagger} e^{i\frac{n\pi}{L}x^-}\right]. \tag{2}$$

The normal ordered Hamiltonian is given by

$$\begin{aligned}
H = {}& -\mu^2\sum_n \frac{1}{n}a_n^{\dagger}a_n + \frac{\lambda}{4\pi}\sum_{k\le l,m\le n} \frac{1}{N_{kl}^2}\frac{1}{N_{mn}^2}\frac{1}{\sqrt{klmn}}a_k^{\dagger}a_l^{\dagger}a_n a_m\ \delta_{k+l,m+n} \\
& +\frac{\lambda}{4\pi}\sum_{k,l\le m\le n} \frac{1}{N_{lmn}^2}\frac{1}{\sqrt{klmn}}\left[a_k^{\dagger}a_l a_m a_n + a_n^{\dagger}a_m^{\dagger}a_l^{\dagger}a_k\right]\ \delta_{k,l+m+n}
\end{aligned} \tag{3}$$

with

$$\begin{aligned}
N_{lmn} ={}& 1,\ l \ne m \ne n, \\
={}& \sqrt{2!},\ l = m \ne n,\ l \ne m = n, \\
={}& \sqrt{3!},\ l = m = n, \tag{4}
\end{aligned}$$

and

$$\begin{aligned}
N_{kl} ={}& 1, k \ne l, \\
={}& \sqrt{2!},\ k = l. \tag{5}
\end{aligned}$$

We diagonalize this Hamiltonian in the basis of all many-boson configurations at a fixed $K$ where $K$ is the sum of the values of the dimensionless momenta of all bosons in the

| even sector | | odd sector | |
| --- | --- | --- | --- |
| K | dimension | K | dimension |
| 16 | 118 | 15 | 90 |
| 24 | 793 | 23 | 632 |
| 32 | 4186 | 31 | 3431 |
| 40 | 18692 | 39 | 15613 |
| 50 | 102162 | 49 | 86809 |
| 60 | 483338 | 59 | 416006 |
| 64 | 870953 | 64 | 870677 |
| 68 | 1544048 | 68 | 1543687 |

TABLE I: Dimensionality of the Hamiltonian matrix in even and odd particle sectors with periodic boundary condition.

configuration. The Hamiltonian is symmetric under $\phi \rightarrow -\phi$ and thus, with PBC, the Hamiltonian matrix becomes block diagonal in even and odd particle number sectors. The dimensionality of the matrix in the even and odd sectors for different $K$ is presented in Table I. Data in Table I agrees with the earliest work [8] on two dimensional $\phi^4$ theory in DLCQ which presented results for $K(\leq 16)$. We obtained our results on clusters of computers ($\leq 28$ processors) using the Many Fermion Dynamics (MFD) code adapted to bosons [9]. The Lanczos method is used in a highly scalable algorithm.

Since we dropped the $P^+ = 0$ mode, degenerate vacuum states, characterized by a spatially uniform field expectation value, are not explicitly present in our formulation. However, we expect degeneracy of the energy levels in the even and odd particle sectors at sufficiently high resolution, $K$. The argument is as follows. For small coupling, variational coherent states $|\alpha\rangle$ represent a good approximation of the lowest lying physical states [2]. Even and odd states are linear combinations of $|\alpha\rangle$ and $|-\alpha\rangle$ and for large enough $K$ they have the same energy. We also expect that our lowest state will be an excitation above the vacuum state and we will show it corresponds to a configuration with properties of a kink-antikink pair. In Fig. 1 we present a ratio, the difference between the lowest eigenvalues of different parity divided by the classical vacuum energy density, as a function of the inverse resolution. Curves for $\lambda = 0.5, 1.0, 1.5$ all demonstrate the trend to degeneracy in the continuum limit
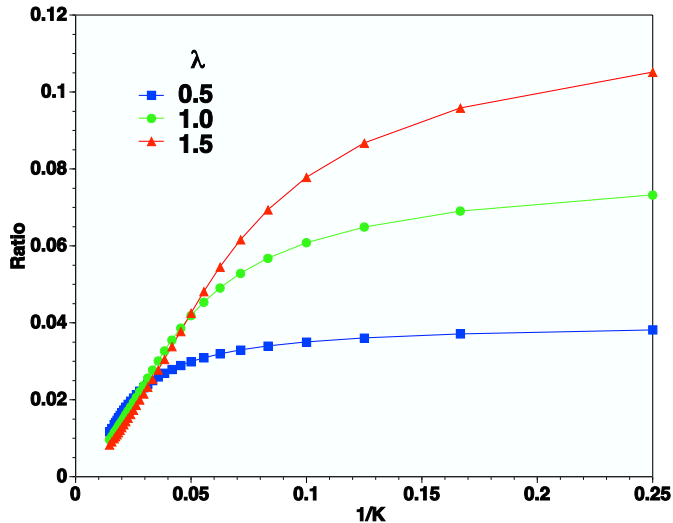
FIG. 1: Ratio of lowest state even-odd energy difference to classical energy density for $\lambda = 0.5, 1.0, 1.5$ .

$(K \to \infty)$. That is, we obtain SSB at each coupling through degeneracy of even and odd parity states when we extrapolate to the continuum limit. At any finite $K$ the lifting of the degeneracy is simply a reflection of the tunneling present in a finite system. As seen from Fig. 1, the tunneling is relatively strong for $K \leq 20$.

Now consider the behavior of the lowest four eigenvalues with $K$ for $\lambda = 0.5, 1.0$ as presented in Fig. 2. For $20 \leq K \leq 72$ we display the even particle number results at even $K$ (which are lower than the odd particle results at the same $K$) and the odd particle number results at odd $K$. The results follow smooth curves becoming more linear as $K$ increases as seen by the straight lines from fitting the $50 \leq K \leq 72$ results.

Our lowest state is expected to be a kink-antikink configuration and should have a positive invariant mass (twice the mass of the single kink at weak coupling). For massive states, the light front energy $E$ scales like $(1/K)$ so that it approaches zero in the infinite $K$ limit. On the other hand, a coherent state variational calculation shows that in the infinite $K$ limit, the energy of the lowest state approaches the classical ground state energy density $\mathcal{E} = -(6\pi\mu^4/\lambda)$ for small $\lambda$. As we show in what follows, our results are increasingly compatible with a coherent state as $K$ increases towards the continuum limit. Thus, we fit our finite $K$ results for the eigenvalues at small $\lambda$ to the formula $C + M^2/K$, where $C$ is
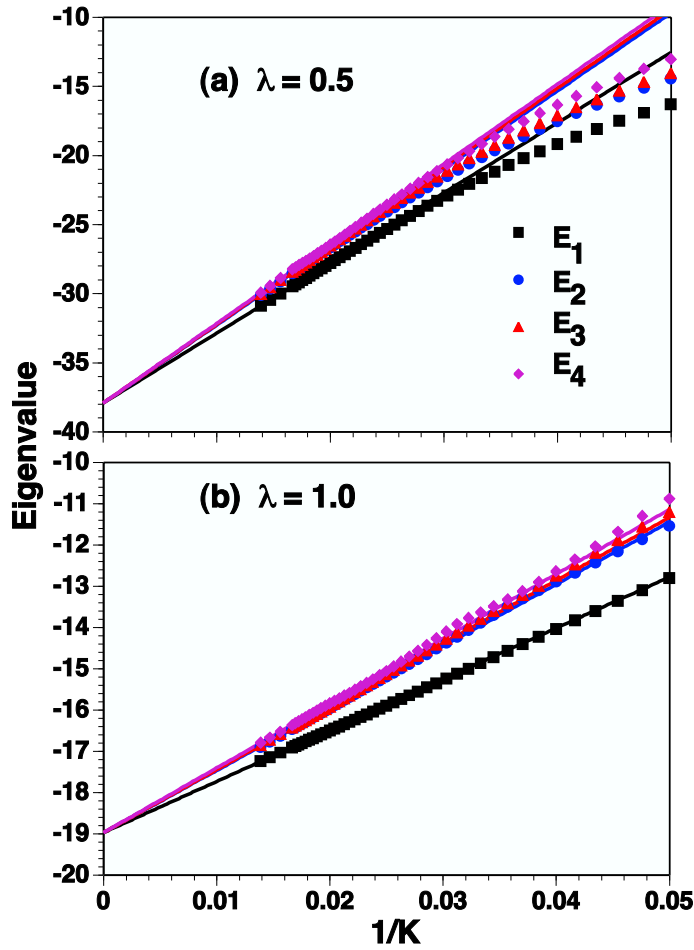
FIG. 2: Lowest four eigenvalues versus $\frac{1}{K}$ for (a) $\lambda = 0.5$. (b) $\lambda = 1.0$.

the vacuum energy density and $M$ is the kink-antikink mass. Extracted values of $C$ and

| $\lambda$ | vacuum energy | | soliton mass | | |
|---|---|---|---|---|---|
| | classical | this work | classical | semi-class. | this work |
| 0.5 | -37.70 | -37.90(4) | 11.31 | 10.84 | 11.26(4) |
| 1.0 | -18.85 | -18.97(2) | 5.657 | 5.186 | 5.563(7) |

TABLE II: Comparison of vacuum energy and soliton mass from the continuum limit of our results, with classical results. Semi-classical results for the mass [3] are also shown. Our estimated uncertainties in the last significant digit are quoted in parenthesis.

the kink or soliton mass $M/2$ are compared to their classical and one loop corrected ("semi-class.") counterparts in Table II. Uncertainties, quoted in parenthesis, are estimated from
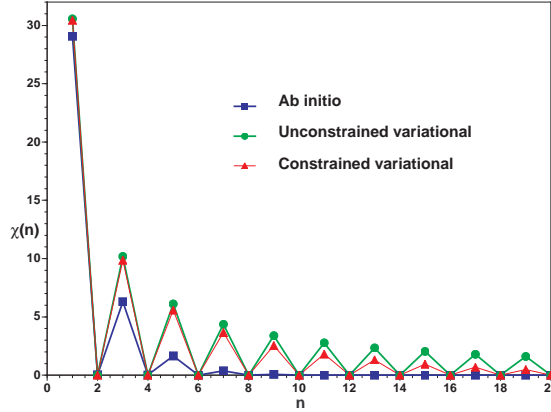
FIG. 3: Comparison of the number density $\chi(n)$ from our approach ("Ab initio") ($K = 60$) and the constrained and unconstrained coherent state variational calculation for $\lambda = 1.0$.

experience after performing a variety of extrapolations. Our vacuum energy nearly coincides with the classical result probably as the result of dropping the zero mode. $M/2$ is also close to the classical value since our coupling constant is small and the kink-antikink interactions are weak. Our soliton mass and vacuum energy at $\lambda = 1$ are in reasonable agreement with results using antiperiodic boundary conditions [10]. The fact that the mass of the quantum kink is larger than the semi-classical one is peculiar to the choice $\mu^2 = 1$ and does not occur for $\mu^2$ away from 1 [6]. Using similar fits to the next excited state, constrained to have the same $1/K \to 0$ intercept (vacuum energy), we obtain the $M^2$ gap. For $\lambda = 0.5$ (1.0) it is 57.3 (26.6). These results are approximately 89% (83%) of the corresponding variational estimates of Ref. [2].

As an example of another observable, we evaluate the occupation number density $\chi(n)$, the analog of the parton distribution function of more realistic theories. Note that in the unconstrained variational state [2], the shape of $\chi(n)$ is independent of the coupling $\lambda$ which affects only its overall normalization. On the other hand, in the variational calculation, constrained to have a fixed value of $\langle K \rangle$, $\lambda$ affects not only the overall normalization but also the shape of the distribution. In Fig. 3 our result at $K = 60$ is compared with that of the unconstrained and constrained ($\langle K \rangle = 60$) coherent state approximation for $\lambda = 1.0$. We find that the shape and normalization of our $\chi(n)$ depends on $\lambda$. Our results display the same sawtooth pattern as the constrained and unconstrained variational results. This is due to the PBC and the sawtooth pattern is not present in the number density in the case of

antiperiodic BC [10]. We do expect sensitivity to the boundary conditions in topologically non-trivial sectors even in the infinite volume limit.

Another observable that yields information about the spatial structure of the low lying states is the Fourier transform of the form factor. We compute this observable for the lowest state which, according to Goldstone and Jackiw [4], represents the classical kink-antikink profile in the weak coupling limit. Let $\mid K \rangle$ and $\mid K' \rangle$ denote this state with momenta $K$ and $K'$. In the continuum,

$$\int_{-\infty}^{+\infty} dq^+ exp\{-\frac{i}{2}q^+a\}\langle K \mid \Phi(x^-) \mid K' \rangle = \phi_c(x^- - a).$$

For a study of the form factor of topological objects in the semi-classical approximation at finite volume with relevance to the work of Goldstone and Jackiw, see Ref. [11].

For the form factor, we require the same state at different $K$ values since $K' = K + q$. We proceed as follows. We diagonalize the Hamiltonian, say, at $K = 40$ in the even particle sector. Then we diagonalize the Hamiltonian at the neighboring odd $K$ values, $K = 31, 33, \ldots 49$ in the odd particle sector so that the dimensionless momentum transfer ranges from -9 to 9. In the sum replacing the above integral in $q^+$, we employ states that fall on the same linear trajectory in Fig. 2 so that we can be confident that all these states correspond to the same physical state. We then compute the matrix element of the field operator between the lowest state at $K = 40$ and all other values of K and sum the amplitudes with the shift parameter $a = 0$. Here, we need to be careful about the phases. First we note that $K$ is a conserved quantity, so eigenfunctions at different $K$ have an overall arbitrary complex phase factor. To set the relative phases between pairs of eigenstates at different $K$, we set the signs of the leading amplitude to be the same as dictated by the coherent state variational calculation. The topology of a kink - antikink structure and other properties of the form factor rely on the detailed behavior of the amplitudes over a range of $K$ values. The result for the lowest eigenstate for $\lambda = 1$ is presented in Fig. 4. This striking kink - antikink behavior is particular to the lowest state.

Taking the vacuum energy results of Table II together with the $\phi_c(x^-)$ results of Fig. 4, we have two independent but consistent methods for extracting $\langle \phi \rangle$ in the weak coupling limit. From our vacuum energy density for $\lambda = 1.0$, we obtain $\langle \phi \rangle = (\mathcal{E}/\pi\mu^2)^{1/2} = 2.457$. From the calculation of the profile function, shown in Fig. 4, we extract $\langle \phi \rangle$ as the asymptotic $(x^- = \pm 1$ in units of $L$) intercepts to be equal to 2.447. These results may be compared
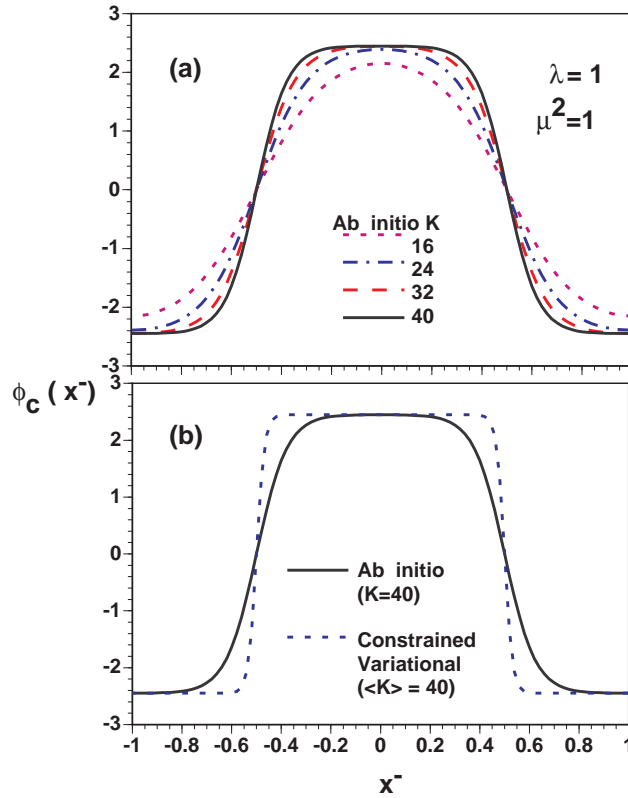
FIG. 4: Fourier Transform of the kink-antikink form factor at $\lambda = 1.0$. Results are plotted in units of $L$. (a) Convergence with $K$. (b) Comparison of our result (Ab initio) ($K$=40) with constrained variational calculation ($\langle K \rangle = 40$).

with the classical value of $\sqrt{6} = 2.449$, which agrees with the result from the variational coherent state.

In summary, we have demonstrated the phenomenon of spontaneous symmetry breaking in a discretized light front approach without $P^+$ zero mode and calculated several nonperturbative physical quantities. We find that a finite Fock space yields features of the lowest excitation that are similar to those of a variational coherent state ansatz. We have extracted the quantum kink mass and the vacuum energy density for small $\lambda$ by extrapolating our lowest eigenvalue to the continuum limit. At weak coupling, the mass of the quantum kink is closer to the classical value than to the semi-classical mass. We have extracted the number density of elementary constituents of the lowest state and compared it with the coherent state prediction. We have also evaluated the Fourier transform of the lowest state form factor in a fully non-perturbative quantum approach and obtained a kink-antikink profile.

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